Frequency Domain Packet Scheduling with Stability Analysis for 3GPP LTE Uplink

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Abstract—In this paper, we investigate the Frequency Domain Packet Scheduling (FDPS) problem for 3GPP Long Term Evolution (LTE) Uplink (UL). Instead of studying a specific scheduling policy, we provide a unified approach to tackle this issue. First, we formalize a general LTE UL FDPS problem, which is suitable for various scheduling policies. Then, we prove that the problem is MAX SNP-hard, which implies that approximation algorithms with constant approximation ratios are the best that we can hope for. Therefore, we design two approximation algorithms, both of which have polynomial runtime. The first algorithm is based on a simple greedy method. The second one is based on the *Local Ratio* (L-R) technique and it can approximately solve the LTE UL FDPS problem with an approximation ratio of 2. To further analyze the stability of the 2-approximation L-R algorithm, we derive a specific FDPS problem, which incorporates the queue length and channel quality information. We utilize the *Lyapunov Drift* to prove the L-R algorithm is stable for any (ω_0, ϵ_0)-admissible LTE UL systems. The simulation results indicate good performance of the L-R scheduler.

Index Terms—Long term evolution (LTE), uplink (UL), frequency domain packet scheduling (FDPS), Optimization Algorithm, approximation ratio, local ratio, stability analysis, Lyapunov drift, (ω, ϵ)-admissible

1 Introduction

The Third Generation Partnership Project (3GPP) Long Term Evolution (LTE) standardization is the next forward step in cellular network services. The objective of LTE is to achieve a high peak-data-rate that scales with scalable system bandwidths, system capacity and coverage improvements, spectrum efficiency, latency reduction, and packet optimized radio access [2]. An architecture of LTE network is depicted in Fig. 1. Its functionality is divided into three main domains: User Equipment (UE), Evolved UMTS Terrestrial Radio Access Network (EU-TRAN), and System Architecture Evolution (SAE) core, also known as Evolved Packet Core (EPC). EUTRAN, which consists of enodeBs, is similar in function to the combination of nodeB and radio network controller (RNC) in the traditional UTRAN. This simplification is beneficial to reduce the latency of all radio interface operations. The eNodeBs are connected by the X2 interface, and they connect to EPC networks using the S1 interface. The EPC network serves as the equivalent GPRS networks via three components, i.e., Mobility Management Entity (MME), Serving Gateway (SGW), and PDN Gateway (PGW). The MME provides tracking and paging for idle mode UEs. The SGW provides switching and routing for user data packets. The PGW provides access to external networks, such as the Internet.

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Because of the robustness against multipath fading, higher spectral efficiency, and bandwidth scalability, the Orthogonal Frequency Division Multiplexing Access (OFDMA) has been selected for the LTE Downlink (DL) [3]. However, OFDM has a high Peak-to-Average Power Ratio (PAPR), which is undesirable and inefficient for the UE terminal and drains the battery very fast [4], [5]. Thus a precoded version of OFDM, the Single-Carrier Frequency Division Multiplexing Access (SC-FDMA) is selected for the LTE Uplink (UL) [5], [6], [7]. SC-FDMA retains the multipath resistance and flexible subcarrier frequency allocation offered by OFDM. Moreover, it has a significantly low PAPR like traditional single-carrier formats such as GSM, which compensates for the drawback with normal OFDM. For the UL, LTE provides the user speed up to 50 Mbps in a 20-MHz channel [5].

Both in the LTE DL and UL, the system bandwidth is divided into separable chunks denoted as Resource Blocks (RBs) (see Fig. 1). An RB is considered as the minimum scheduling resolution in the time-frequency domain. The Frequency Domain Packet Scheduling (FDPS) allocates different RBs to individual users according to their current channel conditions, queue lengths, and other information. The FDPS policy is conducted during each Transmission Time Interval (TTI, in LTE, 1TTI = 1 ms). The UL SC-FDMA is achieved by the FDPS assignment of different frequency portions of bandwidth, which simultaneously realizes frequency-domain multiplexing in concert with time-domain scheduling.

The FDPS algorithm plays an important role for the system performance of LTE. Unfortunately, the excellent scheduling policies in conventional cellular networks can be hardly adopted in the LTE UL system. Most of all, SC-FDMA

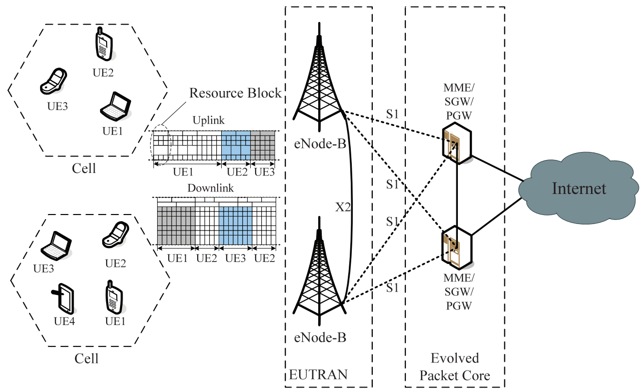

Fig. 1. The architecture of 3GPP LTE network.

induces the *continuous allocation constraint* for the FDPS [8]. Namely, the LTE UL FDPS requires all the RBs allocated to a single user must be contiguous in frequency domain within each TTI (see the Uplink in Fig. 1), because the underlying waveform of SC-FDMA is essentially single carrier [6]. Hence, it becomes harder to perform appropriate resource scheduling for the LTE uplink.

The LTE UL FDPS problem has been addressed in existing literature. Lim et al. [9] propose two utility-based scheduling schemes for SC-FDMA systems. Afterwards, this work was improved by taking the delayed channel state information into consideration [10]. Ruiz de Temino et al. [11] propose three channel-aware scheduling algorithms to address the localized RB allocation. Lee et al. [12] formalize the proportional-fair (PF) FDPS problem (PF-FDPS) for the LTE uplink as a combinatorial optimization problem and prove that the problem is NP-hard. They propose four approximation algorithms based on heuristics and conduct an extensive simulation study on the performance of the four heuristic algorithms. The performance of three distinct schedulers for LTE uplink are compared in [13]. However, it mainly focuses on the impact of flow-level dynamics resulting from the random user behavior.

However, the selection of the scheduling policy for a specific LTE UL system depends on a case by case analysis, which is not the focus of our work. In this paper, rather than a particular scheduling policy, we present a unified approach to tackle the LTE UL FDPS problem. We define the *profit function* that indicates the profit gained by allocating a set of contiguous RBs to an active user. The profit function is capable of expressing various scheduling objectives, including the PF metric [12] and scheduling policies that combine utility maximization and queue stability [14]. Based on the profit function, we formalize a general FDPS problem for the LTE UL that can cover many LTE UL scheduling objectives.

Furthermore, we address the hardness of the LTE UL FDPS problem. We prove that the scheduling problem is MAX SNP-hard, which means that the problem is improbable to efficiently approximate within a certain ratio. Accordingly, the approximation algorithms with constant approximation ratios are the best solutions that we can hope for. Subsequently, we design two approximation algorithms in company with provable approximation ratios, which solve the LTE UL FDPS problem in polynomial runtime. The first algorithm is based on a greedy method. It is intuitive and easy to follow, but its approximation ratio is bounded by a slowly increasing function of the number of active users. The second one is designed on the basis of the *Local Ratio* technique [15], and is called *L-R Algorithm* hereafter. Although the *L-R Algorithm* is relatively sophisticated, it achieves a constant approximation ratio of 2.

The assumption of an infinitely backlogged model in scheduling analysis is conventional in many literatures, which implies there are always arrival packets to serve. However, this is not always the case in practical systems. In real networks, packets are generated for each user in accordance with a stochastic arrival process with respect to applications. Andrews [16] makes a survey of scheduling theory in wireless networks with system stability considerations in cases of infinite and finite backlog. In the follow-on work [14], Andrews and Zhang analyze the multicarrier situation, where finite queues are fed by an admissible arrival process, and point out the PF scheduler does not work so well. In particular, it can result in the instability of queues [17]. Provided that the schedulers have no idea of the queues length, they may even schedule an empty queue though its channel quality is good. Meanwhile, the queue full of packets may have no chance to send once it faces a bad channel condition. In those two unfavorable situations, the queues are not stable in a bounded length. Therefore, it is preferable to take system utility maximization and queuing stability into account simultaneously when we design scheduling algorithms. Besides, the well-known *MaxWeight* algorithm is generalized from the single-carrier settings to accommodate a number of natural optimization problems in the multicarrier settings [14]. Moreover, the hardness of these problems are stated and the simple algorithmic solutions are designed with provable performance bounds, which exactly keeps the queuing system stable.

Since the *L-R Algorithm* achieves a fairly good performance, it is worthy to investigate the stability of the algorithm. We specify the *profit function* to incorporate the queue length and channel quality information, and thus formalize a specific LTE UL FDPS problem. Because the specific problem is a special case of the general problem, the *L-R Algorithm* can also be applied with the same approximation ratio of 2. Similar to the methodology used in [14], we utilize the *Lyapunov Drift* to prove the queuing stability for the *L-R Algorithm* in case of finite buffer and arbitrary arrival process. First, we extend the definition of (ω, ϵ)-admissible system to the LTE UL multi-carrier scenario. The (ω, ϵ)-admissible explicitly describes the basic propositions of some LTE UL systems, which will be stably scheduled by the feasible FDPS algorithms. We assume that the optimal scheduler for LTE UL FDPS is stable for any (ω, ϵ)-admissible LTE UL systems. Then, the *Lyapunov drift* is computed and the queuing stability of the *L-R Algorithm* is proved in any (ω_0, ϵ_0)-admissible LTE UL systems. Finally, The values of ω_0 and ϵ_0 are calculated.

The remainder of the paper is organized as follows: Section 2 introduces the necessary background in the theory of approximation and complexity. Section 3 gives the system model we study and formalizes the LTE UL FDPS problem. Section 4 proves the LTE UL FDPS problem is

MAX SNP-hard and shows polynomial time approximation algorithms with guaranteed approximation ratios are necessary for practical LTE UL systems. In Section 5 and 6, respectively, we give two approximation algorithms computable in polynomial time and find out their approximation ratios. Section 7 analyzes the queuing stability of the 2-approximation *L-R Algorithm*. The performance of the *L-R Algorithm* is evaluated in Section 8. Finally, we conclude the paper in Section 9.

2 Preliminaries

In this section, we provide a brief introduction to the theory of both optimization and stability used in this work.

2.1 Approximation Ratio of an Approximation Algorithm

In practice, many optimization problems are NP-hard, which means that exact solutions of these problems are widely believed to be time consuming. This necessities efficient approximation algorithms that always compute solutions close to the optimum. However, different approximation algorithms can achieve different degrees of approximation for the same problem. So, we introduce the definition of approximation ratio to qualify the degree of approximation achieved by an approximation algorithm.

Assume that G is an instance of a maximization problem.[1] We denote the size of its input by $|G|$ and its optimal value by $OPT(G)$. Let ALG be an approximation algorithm for the maximization problem. For instance G, we denote the value of ALG by $ALG(G)$. We say that ALG has an approximation ratio of $\rho(|G|)$ [18] if, for any instance G, $OPT(G)$ is within a factor of $\rho(|G|)$ of $ALG(G)$:

$$OPT(G) \leq \rho(|G|) \cdot ALG(G).$$

We also call ALG a $\rho(|G|)$-approximation algorithm. When the approximation ratio is independent of the input size $|G|$, we will use the terms approximation ratio of ρ and ρ-approximation algorithm, indicating no dependence on $|G|$.

Note that $\rho(|G|)$ (or ρ) always ≥ 1, and a smaller value of $\rho(|G|)$ (or ρ) indicates that the approximation algorithm has a better performance in a worst-case sense. In particular, when $\rho = 1$, the approximation algorithm ALG essentially finds the optimal solution for any instance G.

2.2 Polynomial-Time Approximation Scheme (PTAS)

A polynomial-time approximation scheme [18] for a maximization problem is an approximation algorithm that takes as an input not only an instance of the problem, but also a value $\epsilon > 0$ such that for any fixed ϵ, the scheme is a $(1+\epsilon)$-approximation algorithm, which is computable in polynomial time in the size of the input instance.

In a technical sense, a PTAS is the best one can hope for an NP-hard optimization problem, assuming $P \neq NP$.

2.3 L-Reduction

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are maximization problems. An L-reduction [19] from $\mathcal{A}$ to $\mathcal{B}$ is a pair of functions R and S, both computable in polynomial time, with the following two additional properties:

First, if X is an instance of $\mathcal{A}$ with optimum $OPT(X)$, then $R(X)$ is an instance of $\mathcal{B}$ with optimum $OPT(R(X))$ that satisfies

$$OPT(R(X)) \leq \alpha \cdot OPT(X), \quad (1)$$

where α is a positive constant.

Second, if s is any feasible solution of $R(X)$, then $S(s)$ is a feasible solution of X such that

$$OPT(X) - VAL(S(s)) \leq \beta \cdot (OPT(R(X)) - VAL(s)), \quad (2)$$

where β is another positive constant particular to the reduction and VAL denotes the value of the feasible solution in both instances. Equation (2) guarantees that S returns a feasible solution of X, which is not much more suboptimal than the given solution of $R(X)$. In particular, if s is the optimal solution of $R(X)$, then $S(s)$ is the optimal solution of X.

L-reductions have the composition property [19]:

Lemma 1. *If (R, S) is an L-reduction from problem $\mathcal{A}$ to problem $\mathcal{B}$, and (R', S') is an L-reduction from problem $\mathcal{B}$ to problem $\mathcal{C}$, then their composition $(R \cdot R', S' \cdot S)$ is an L-reduction from $\mathcal{A}$ to $\mathcal{C}$.*

2.4 MAX-SNP Hardness

In computational complexity theory, SNP (from Strict NP) is a complexity class containing a limited subset of NP based on its logical characterization in terms of graph-theoretical properties. The class MAX SNP is a subset of optimization problems derived from SNP. The formal definition of MAX SNP can be found in [19]. A problem is said to be MAX SNP-hard if all MAX SNP problems can be L-reduced to this problem. MAX SNP-hard problems are hard to approximate. It is shown in [19] that

Lemma 2. *Any MAX SNP-hard problem does not have a PTAS unless $P = NP$.*

Suppose $\mathcal{A}$ is a known MAX SNP problem, thus all the MAX SNP problems can be L-reduced to $\mathcal{A}$. Once $\mathcal{B}$ can be L-reduced from $\mathcal{A}$, according to the composition property in Lemma 1, all the MAX SNP problems can also be L-reduced to $\mathcal{B}$, which indicates $\mathcal{B}$ is MAX SNP-Hard. In other words, to prove that a problem $\mathcal{B}$ is MAX SNP-hard, it suffices to present an L-reduction from a known MAX SNP-hard problem $\mathcal{A}$ to $\mathcal{B}$.

2.5 Job Interval Selection Problem with k Intervals per Job

The job interval selection problem with k intervals per job (JISPk) is stated as follows [20]:

Input. We are given n job, each of which is associated with k intervals on the real line. Thus, we have $k \cdot n$ intervals. For each interval l a starting time s_l and a finishing time $f_l(>s_l)$ is known, $l = 1, \ldots, k \cdot n$. All starting and finishing times are integers. An interval is said to be active at time t if and only if $t \in [s_l, f_l)$. Two interval intersect if and only if there is a time t during which both intervals are active.

1. We focus on maximization problems in this paper because LTE UL FDPS is a maximization problem.

Goal. Select as many intervals as possible such that 1) for each job, at most one interval is selected from its associated k intervals, and 2) no two selected intervals intersect.

Measure. The number of intervals selected.

Spieksma [20] presents an L-reduction from MAX-3SAT-B to JISP2. Since MAX-3SAT-B is MAX SNP-hard [19], according to Lemma 1, [20] essentially proves that

Lemma 3. *JISP2 is MAX SNP-hard, and hence it does not have a PTAS unless* $P = NP$.

In the JISPk, k is a parameter of problem. In this paper, we refer to the JISP2 problem, namely, $k = 2$.

2.6 Local Ratio for Scheduling Problems

The local ratio technique is a methodology for the design and analysis of algorithms in a broad range of optimization problems. It is first developed by Bafna et al. [21], later extended by Bar-Yehuda [22], both of which treat minimization covering problems. Bar-Noy et al. [15] present the first local ratio algorithm for maximization problems. Bar-Yehuda and Rawitz, [23] develop a novel extension of the technique, called emphfractional local ratio.

Local ratio can be utilized to solve the profit maximization problem, that is, given a profit vector $\mathbf{p} \in \mathbb{R}^n$, to find a solution vector $\mathbf{x}$ that maximizes the inner product $\mathbf{p} \cdot \mathbf{x}$, subject to a given set $\mathbb{F}$ of feasibility constraints on $\mathbf{x}$. The fundamental Local Ratio Theorem for maximization problems is presented next, with proof omitted.

Theorem 1 (Local ratio). *Let* $\mathbb{F}$ *be a set of constraints and let* p, p_1, *and* p_2 *be profit vectors such that* $p = p_1 + p_2$. *Then, if* x *is an r-approximation solution with respect to* $(\mathbb{F}, p_1)$ *and with respect to* $(\mathbb{F}, p_2)$, *it is an r-approximation solution with respect to* $(\mathbb{F}, p)$.

A massive applications of the technique are solving the *Machine Scheduling* problem. The resource consists of k parallel machines and the activities are jobs to be scheduled on these machines. Each job can be scheduled in one of several time intervals. The goal is to maximize the profit of the executed jobs. There are several subcases to this problem, including Interval Scheduling, Independent Set in Interval Graphs, k-Colorable Subgraph, Parallel Unrelated Machines, and Scheduling with Release Times and Deadlines. Besides, the technique can be used for *Bandwidth Allocation*. The problem aims at finding the most profitable set of sessions that can utilize the available bandwidth. A 5-approximation algorithm based on local ratio is designed to tackle it. The above problems are all for profit maximization, which is similar to the LTE UL FDPS problem that we focus in this paper. Due to the page limits, other applications of local ratio, for example, the *Loss Minimization* and the detailed survey can be found in [24].

2.7 Lyapunov Stability Analysis

According to [25], the definition of system stability and its sufficient conditions are presented. Suppose the state space is partitioned in the sets $T, R_1, R_2 \ldots$, where $R_j, j = 1, 2 \ldots$, are closed sets of communicating states and T contains all states that do not belong to any closed set of communicating states and therefore are transient.

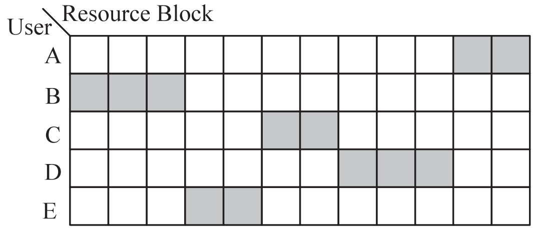

Fig. 2. A feasible FDPS scheduling for the LTE uplink. $m = 12, n = 5$. The shadow (user, RB) pairs denote the RB-to-user assignment of the feasible schedule. Note that the RBs assigned to each user must be contiguous in frequency domain.

Definition 1. *The system is stable if for the queue length process* $\mathcal{Y}$, *we have*

$$P(\tau_y < \infty) = 1 \quad \forall y \in T,$$

where τ_y *is the recurrent time and all states* $y \in \cup_{j=1}^{\infty} R_j$ *are positive recurrent.*

Theorem 2 states the sufficient conditions for system stability based on Definition 1. Those conditions involve the drift of a Lyapunov function on the state space of a Markov chain.

Theorem 2. *Consider a Markov chain* $M(t)$ *with state space* $\mathcal{M}$. *If there exists a lower bounded real function* $V : \mathcal{M} \to \mathbb{R}$, $\phi > 0$ *and a finite subset* $\mathcal{M}_0$ *of* $\mathcal{M}$ *such that*

$$E[V(M(t+1)) - V(M(t)) \mid M(t) = y] \le -\phi \ \text{ if } y \notin \mathcal{M}_0$$

$$E[V(M(t+1)) \mid M(t) = y] < \infty \ \text{ if } y \in \mathcal{M}_0,$$

then for the time τ_y *in Definition 1, we have*

$$P(\tau_y < \infty) = 1 \quad \forall y \in T,$$

and all states $y \in \cup_{j=1}^{\infty} R_j$ *are positive recurrent.*

The detailed introduction and formal definition of the Lyapunov stability theory can be found in [26].

3 Problem Formalization

3.1 System Model

We consider the uplink of cellular network, where the system bandwidth is divided into m RBs. Besides, the network has a single base station and n active wireless users. We denote the set of all RBs by M ($M = \{1, 2, \ldots, m\}$) and the set of all users by N ($N = \{1, 2, \ldots, n\}$). Within each TTI, the FDPS allocates m RBs to n users in a *contiguous fashion*. Namely, a set of contiguous RBs is distributed to each user while the empty set of RBs to a user means he is not scheduled in this round. Specifically, each RB is assigned to at most one user. Fig. 2 illustrates an example of a feasible FDPS scheduling for the LTE UL.

We denote by A the collection of all sets of contiguous RBs, where $A \subseteq \mathcal{P}(M)$. $\mathcal{P}(M)$ represents the power set of M, i.e., the collection of all subsets of M and $\forall a \in A$, $a = \{c, c+1, \ldots, c+l\}$, $1 \le c \le c+l \le m$. For $a, b \in A$, we call a intersects b if $a \cap b \neq \emptyset$. $\forall a \in A$, we use $head(a)$ to denote the smallest RB of a, and $tail(a)$ the largest RB of a. The Boolean variable x_i^a is used to indicate whether or not the set of contiguous RBs a is assigned to user i. User i gets the set $a \in A$ if and only if $x_i^a = 1$.[2]

We define the profit function as follows:

$$p(a, i) : A \times N \rightarrow \mathbb{R},^{\geq 0^2}$$

where $p(a, i)$ indicates the profit gained by assigning $a \in A$ to user i in a feasible schedule (in one TTI).

The profit function $p(a, i)$ is a general term and might vary with different schedulers. We can exploit it to model various specific scheduling algorithms. For example, $p(a, i) = \sum_{c \in a} \lambda_i^c$ is the Proportional Fair scheduling objective studied in [12], where λ_i^c is the PF metric value that user i has on RB c. In addition, we can, respectively, express the three objective functions in [14] as

$$p(a, i) = Q_i \sum_{c \in a} R_i^c, \tag{3}$$

$$p(a, i) = Q_i \min\left\{Q_i, \sum_{c \in a} R_i^c\right\}, \tag{4}$$

$$p(a, i) = (Q_i)^2 - \left(\max\left\{0, Q_i - \sum_{c \in a} R_i^c\right\}\right)^2, \tag{5}$$

where Q_i is the queue size for user i at the beginning of each TTI, and R_i^c is the data rate for user i, RB c in this TTI. These three objective functions combine throughput maximization and queue stability together. In Section 7, we specify $p(a, i)$ as (3) to incorporate the queue length and channel quality information, and thus formalize a specific FDPS problem to analyze the stability of our proposed scheduling algorithm.

3.2 LTE UL FDPS

We consider a general FDPS problem for the LTE uplink. We are given an uplink system with m RBs and n users. In one time slot, for each set of contiguous RBs $a \in A$ and each user i, we have a profit $p(a, i)$. Our goal is to schedule the system for this time slot. In other words, we intend to find the most advantageous way to assign an $a \in A$ to user i so that the total profit is maximized. The LTE UL FDPS problem is formalized as the following combinatorial optimization problem:

$$\begin{aligned} &\max \sum_{(a,i) \in A \times N} p(a, i) \cdot x_i^a \\ &\text{subject to :} \\ &\text{for each RB } c \in M : \sum_{i \in N, a: c \in a} x_i^a \leq 1, \\ &\text{for each user } i \in N : \sum_{a \in A} x_i^a \leq 1, \\ &\text{for } i \in N, a \in A : x_i^a \in \{0, 1\}. \end{aligned} \tag{6}$$

The first constraint shows that every RB is assigned to at most one user, and the second constraint ensures that each user can get no more than one set of contiguous RBs. In fact, LTE UL FDPS aims to find a subset of $A \times N$ that maximizes its total profit in a time slot, according to the scheduling policy specified in the profit function. Problem (6) is a binary integer programming and it is not hard to find that the PF-FDPS problem studied in [12] is a special case of (6).

2. $\mathbb{R}^{\geq 0}$ denotes the nonnegative real number set, i.e., $p(a, i) \geq 0$ for all $a \in A, i \in N$.

4 Hardness Results

4.1 Hardness of (6)

It is not difficult to show that LTE UL FDPS is NP-hard. Lee et al. [12] have shown that the LTE UL PF-FDPS problem, a special case of LTE UL FDPS, is NP-hard. Furthermore, it is straightforward to reduce the LTE UL PF-FDPS problem to LTE UL FDPS (by simply setting $p(a, i) = \sum_{c \in a} \lambda_i^c$). Thus, LTE UL FDPS is NP-hard.

Furthermore, we have a stronger result here.

Theorem 3. *LTE UL FDPS is MAX SNP-hard, and hence, it does not have a PTAS assuming $P \neq NP$.*

Proof. We prove this theorem by presenting an L-reduction from JISP2 to LTE UL FDPS.

Assume that X is an instance of JISP2. X has n jobs, and we denote them by $J_1, J_2, \ldots, J_n$. Job J_i has two intervals, namely $[s_i^{(1)}, f_i^{(1)})$ and $[s_i^{(2)}, f_i^{(2)})$. $s_i^{(j)}$ and $f_i^{(j)}$ are integers, and $f_i^{(j)} > s_i^{(j)}$, $j = 1, 2$, $i = 1, 2, \ldots, n$.

Now, we construct function R. $R(X)$ is defined as follows: Let $m = \max_{i=1,2,\ j=1,\ldots n}\{f_i^{(j)}\} - 1$. $R(X)$ is an LTE uplink system, which has n users and m RBs. $N = \{1, 2, \ldots n\}$ is the set of active users. A is the set of all contiguous RBs such that $\forall a \in A$, $a = \{i, i+1, \ldots, i+l\}$, $1 \leq i \leq i+l \leq m$. User i corresponds to job J_i, $i = 1, 2, \ldots, n$, and the profit function $p : A \times N \rightarrow \mathbb{R}^{\geq 0}$ is defined as follows:

$$p(a, i) = \begin{cases} 1 & \text{if } a = \{s_i^{(j)}, s_i^{(j)} + 1, \ldots, f_i^{(j)} - 1\}, \\ & j = 1, 2, \\ 0 & \text{otherwise.} \end{cases}$$

Since $s_i^{(j)}$ and $f_i^{(j)}$ are integers, and $f_i^{(j)} > s_i^{(j)}$, $j = 1, 2$, $i = 1, 2, \ldots, n$, the profit function p is well defined.

Next, we construct function S. Assume that s is a feasible solution of $R(X)$. s can be written as $\{(a_i, u_i) \mid i = 1, 2, \ldots, k, a_i \in A, u_i \in N\}$. We define $p_{supp} = \{(a, i) \mid p(a, i) = 1, (a, i) \in A \times N\}$. Let $s' = s \cap p_{supp}$. Obviously, s' is also a feasible solution of $R(X)$ such that each element of s' has a positive profit. That is, in s', each user at most selects one set of contiguous RBs and no two a_is appearing in s' intersect.

According to the definition of the profit function p, $\forall (a_i, u_i) \in s'$, job J_{u_i} is associated with an interval $[head(a_i), tail(a_i) + 1)$ in X. Therefore, $S(s)$ is defined as

$$S(s) = \{([head(a_i), tail(a_i) + 1), J_{u_i}) \mid (a_i, u_i) \in s'\}.$$

We use the pair $([head(a_i), tail(a_i) + 1), J_{u_i})$ to denote that in $S(s)$, job J_{u_i} selects the interval $[head(a_i), tail(a_i) + 1)$. Since s' is a feasible solution of $R(X)$, correspondingly in $S(s)$, each job at most selects one interval, and no two selected intervals intersect. So, $S(s)$ is a feasible solution of X. Thus, function S is well defined.

Now, we check (1). Assume that s_0 is the optimal solution of $R(X)$, and define $s'_0 = s_0 \cap p_{supp}$. Apparently, we have

$$VAL(s'_0) = VAL(s_0) = OPT(R(X)),$$

where $VAL(s'_0)$ and $VAL(s_0)$ denote the values of solutions s'_0 and s_0, respectively. We know that $S(s_0)$ is a feasible solution of X, and we have

$$VAL(s'_0) = VAL(S(s_0)) \leq OPT(X).$$

Thus, we have $OPT(R(X)) \leq OPT(X)$. So, (1) holds ($\alpha = 1$).

Then, we check (2). Let s be any feasible solution of $R(X)$. Obviously, we have

$$VAL(s) = VAL(s') = VAL(S(s)), \tag{7}$$

where $s' = s \cap p_{supp}$. Assume that r is the optimal solution of X, and $r = \{([s_{u_i}, f_{u_i}), J_{u_i}), i = 1, 2, \ldots k\}$. In $R(X)$, we can correspondingly construct $s = \{(\{s_{u_i}, s_{u_i} + 1, \ldots, f_{u_i} - 1\}, u_i), i = 1, 2, \ldots, k\}$. In s, each user is assigned to at most one set of contiguous RBs, and any two sets of contiguous RBs do not intersect. So, s is a feasible solution of $R(X)$. Moreover, since for $R(X)$,

$$p(\{s_{u_i}, s_{u_i} + 1, \ldots, f_{u_i} - 1\}, u_i) = 1, i = 1, 2 \ldots, k,$$

we have

$$OPT(R(X)) \geq VAL(s) = VAL(r) = OPT(X). \tag{8}$$

Combining (7) and (8), we have

$$OPT(X) - VAL(S(s)) \leq OPT(R(X)) - VAL(s).$$

So, (2) holds ($\beta = 1$).

Thus, (R, S) is an L-reduction from JISP2 to LTE UL FDPS. Since JISP2 is MAX SNP-hard, LTE UL FDPS is also MAX SNP-hard and it does not have a PTAS unless $P = NP$. □

Theorem 3 is somewhat devastating, because the nonexistence of PTAS implies that for some constant $\delta > 0$, there are no polynomial time $(1 + \delta)$-approximation algorithms for LTE UL FDPS unless $P = NP$. That is to say, we could at most hope for approximation algorithms that have constant approximation ratios.[3]

4.2 The Size of Search Space of (6)

Despite the fact that the LTE UL FDPS problem is MAX SNP-hard, one may still be tempted to find the optimal solution by an *exhaustive search*, because this approach does not consume much computation power when the search space is small, and enumerating all feasible schedules is sufficient to find the optimal schedule.

In the following, we calculate the number of feasible schedules for the exhaustive search in an uplink system, which has m RBs and n active users. Assume that in a feasible schedule, k out of n users are assigned to contiguous RBs. These k sets of contiguous RBs are denoted as $a_1, \ldots, a_k$ such that $1 \leq head(a_1) \leq tail(a_1) < head(a_2) \leq tail(a_2) \cdots < head(a_k) \leq tail(a_k) \leq m$. So,

$$1 \leq head(a_1) < tail(a_1) + 1 < head(a_2) + 1 < tail(a_2) + 2 \cdots < head(a_k) + k - 1 < tail(a_k) + k \leq m + k.$$

Thus, there is a 1-1 correspondence between the number of choices of k sets of contiguous RBs and the number of $2k$ integers $\{b_i, i = 1, 2, \ldots, 2k\}$ such that $1 \leq b_1 < \cdots < b_{2k} \leq m + k$. So, the number of choices of k sets of contiguous RBs is $\binom{m+k}{2k}$. After assigning k users to k sets of contiguous RBs, we have $\binom{m+k}{2k} \cdot \frac{n!}{(n-k)!}$ feasible schedules in which k active users are assigned to contiguous RBs. So, the total number of feasible schedules is

$$\sum_{k=0}^{n} \binom{m+k}{2k} \cdot \frac{n!}{(n-k)!} > \binom{m+n}{2n} \cdot n! \tag{9}$$

In practical systems, the running time of an exhaustive search is evidently unacceptable for one TTI (in LTE, 1TTI = 1 ms). In other words, the hardness result and the giant search space of (6) imply that approximation algorithms computable in polynomial time with guaranteed performance is indispensable for the LTE UL FDPS problem. In the subsequent sections, we present two polynomial time approximation algorithms. The first one is intuitive and easy to follow, but its performance declines slightly as the number of active users grows. The second one is relatively delicate, but achieves a better performance. The second algorithm is derived from the local ratio technique [15].

5 A Greedy Strategy-Based Algorithm

Here, we present our first approximation algorithm. The main idea of the heuristic algorithm is to divide the LTE UL FDPS problem into several subproblems according to the profit, and then apply a greedy method to each subproblem. We prove that this algorithm has an approximation ratio of $O(\ln n)$, where n is the number of active users in a cell.

5.1 A Special Scheduling Problem

We start from considering a special scheduling problem:

$$\begin{aligned}
&\max \sum_{(a,i) \in \mathcal{C} \subseteq A \times N} x_i^a \\
&\text{subject to:} \\
&\text{for each RB } c \in M: \quad \sum_{(a,i)|(a,i) \in \mathcal{C}, c \in a} x_i^a \leq 1, \\
&\text{for each user } i \in N: \quad \sum_{(a,i) \in \mathcal{C}} x_i^a \leq 1, \\
&\text{for each } (a,i) \in \mathcal{C}: \quad x_i^a \in \{0, 1\}.
\end{aligned} \tag{10}$$

In this problem, a user i may not choose a set of contiguous RBs arbitrarily, but from $\mathcal{C}$, a collection of legitimate sets of (a, i) pairs. All pairs of one user and one of his legitimate sets of contiguous RBs constitute the set $\mathcal{C}$, which is a subset of $A \times N$. In addition, $p(a, i) = 1$, $\forall (a, i) \in \mathcal{C}$. That is, the goal of this problem is to schedule as many users as possible in a time slot. Problem (10) also

3. For PTAS, we can have a $(1 + \epsilon)$-approximation algorithm computable in polynomial time for any $\epsilon > 0$.

complies with the constraints on users and RBs of the LTE UL FDPS problem.

JISPk problem studied in [20] is very similar to (10). Based on [20], we provide a greedy algorithm for (10). $head(a)$ and $tail(a)$ are the smallest and the largest RBs in $a \in A$, respectively.

The main idea of Algorithm 1 is straightforward. It takes as an input $\mathcal{C}$, a collection of admissible pairs of one user and one set of contiguous RBs. The algorithm iteratively selects an (a, i), which has the smallest $tail(a)$ until C is empty. Algorithm 1 outputs G, a feasible schedule of (10). The value of Algorithm 1 is $|G|$. According to a similar Theorem in [20], we can obtain the following theorem. The prove procedure is the same as that in [20], which is therefore omitted here.

Algorithm 1. GREEDY.

input $\mathcal{C}$
$G \leftarrow \emptyset$, $C \leftarrow \mathcal{C}$
while $C \neq \emptyset$ **do**
  $(a^*, i^*) \leftarrow \arg\min_{(a,i)\in C}(tail(a))$ // break ties arbitrary
  $C \leftarrow C \backslash \{(a,i) \mid i = i^*, \text{or } head(a) \leq tail(a^*)\}$
  $G \leftarrow G \cup \{(a^*, i^*)\}$
end while
return G

Theorem 4. *Algorithm 1 is a 2-approximation algorithm for (10). Moreover, it can be shown that the approximation ratio of 2 is tight.*

5.2 The Weighted Version of (10)

Then, we consider the weighted version of (10). We associate each pair $(a, i) \in \mathcal{C}$ with profit $p(a, i)$, and change the objective function to

$$\max \sum_{(a,i)\in\mathcal{C}} p(a,i) \cdot x_i^a. \tag{11}$$

The constraints of (11) are the same as those of (10). This problem is analogous to LTE UL FDPS except that a feasible schedule is selected from a subset $\mathcal{C} \subseteq A \times N$, not from $A \times N$ itself.

We consider the following procedure that provides a feasible solution for (11). For a subset $\mathcal{C} \subseteq A \times N$, we ignore the profit function and run Algorithm 1, and then get a feasible schedule G. Then, we compute $WG = \sum_{(a,i)\in G} p(a,i)$. We denote the optimal value of (11) by $W\text{-}OPT(\mathcal{C})$. The following lemma establishes the relation between WG and $W\text{-}OPT(\mathcal{C})$.

Lemma 4. *If* $\forall (a,i) \in \mathcal{C}$, $0 < m_p \leq p(a,i) \leq M_p$, *then* $W\text{-}OPT(\mathcal{C}) \leq \frac{2M_p}{m_p} \cdot WG$.

Proof. If $\mathcal{C} = \emptyset$, then the lemma holds vacuously. Otherwise, we denote the optimal value of (10) by $S\text{-}OPT(\mathcal{C})$, then $W\text{-}OPT(\mathcal{C}) \leq M_p \cdot S\text{-}OPT(\mathcal{C})$. According to Theorem 4, we have $S\text{-}OPT(\mathcal{C}) \leq 2 \cdot |G|$, so $W\text{-}OPT(\mathcal{C}) \leq 2M_p \cdot |G|$. According to the proposed procedure, we have $WG \geq m_p \cdot |G|$. Combining the above two equations, we finally get $W\text{-}OPT(\mathcal{C}) \leq \frac{2M_p}{m_p} \cdot WG$. □

5.3 The Approximation Algorithm

Taking Algorithm 1 as the subroutine, our first approximation algorithm for the LTE UL FDPS problem is stated as follows (Algorithm 2).

Algorithm 2. GREEDY-BASED (G-B for short).

input m, n, p
$p_{max} \leftarrow \max_{(a,i)\in A\times N} p(a,i)$
partition $A \times N$ into $k+1$ subsets: $S_0, S_1, \ldots, S_k$ such that $S_0 = \{(a,i) \mid p(a,i) \leq \frac{p_{max}}{n}\}$ and $S_j = \{(a,i) \mid \frac{\alpha^{j-1} \cdot p_{max}}{n} < p(a,i) \leq \frac{\alpha^j \cdot p_{max}}{n}\}$ for $j \geq 1$
// $\alpha > 1$ is a constant and $k = \lceil \frac{\ln n}{\ln \alpha} \rceil$. The specification of α will be discussed in the proof of Theorem 5
for $j = 1$ to k **do**
  $G_j \leftarrow GREEDY(S_j)$
  $WG_j \leftarrow \sum_{(a,i)\in G_j} p(a,i)$
end for
$j^* \leftarrow \arg\max_{1\leq j\leq k}(WG_j)$
return G_{j^*} and WG_{j^*}

Algorithm 2 takes m, n, p as the input, where m is the number of RBs, n is the number of active users, and $p : A \times N \rightarrow \mathbb{R}^{\geq 0}$. It first partitions all (a, i) pairs into $k+1$ subsets $\{S_j, j = 0, 1, \ldots, k\}$ according to their profits. For $S_k, k \geq 1$, Algorithm 2 invokes Algorithm 1 to obtain a feasible schedule. Thus Algorithm 2 obtains k feasible schedules, then it chooses the schedule that has the largest total profit as the output.

Theorem 5. *Algorithm 2 is an* $O(\ln n)$*-approximation algorithm for LTE UL FDPS, where* n *is the number of active users in a cell.*

Proof. For any instance $\mathcal{S}$ of (6), we denote the optimal value by $OPT(\mathcal{S})$ and the return value of Algorithm 2 by WG_{j^*}. First, we show that $\frac{OPT(\mathcal{S})}{WG_{j^*}} \leq \alpha + \frac{2\alpha}{\ln \alpha} \ln n$.

In Algorithm 2, all (a, i) pairs are partitioned into $k + 1$ subsets $\{S_j, j = 0, 1, \ldots, k\}$. Each S_j with corresponding profits can be regarded as an instance of (11), whose optimal value is written as $W\text{-}OPT(S_j)$. Then, it is obvious that

$$OPT(\mathcal{S}) \leq \sum_{j=0}^{k} W\text{-}OPT(S_j). \tag{12}$$

For $j \geq 1$, $\frac{\alpha^{j-1} \cdot p_{max}}{n} < p(a,i) \leq \frac{\alpha^j \cdot p_{max}}{n}, \forall (a,i) \in S_j$. So, Lemma 4 indicates that $W\text{-}OPT(S_j) \leq 2\alpha \cdot WG_j, j \geq 1$. For $j = 0$, because the number of users is n, the second constraint of (6) tells us that $W\text{-}OPT(S_0) \leq n \cdot \frac{p_{max}}{n} = p_{max}$. Combining the above two equations, we turn (12) into $OPT(\mathcal{S}) \leq p_{max} + 2\alpha \sum_{j=1}^{k} WG_j$. Note that $\{(a,i) \mid p(a,i) = p_{max}\} \subseteq S_k$, so $S_k \neq \emptyset$. Thus, $WG_k \geq \frac{\alpha^{k-1} \cdot p_{max}}{n}$. So, we have

$$OPT(\mathcal{S}) \leq \frac{n}{\alpha^{k-1}} WG_k + 2\alpha \sum_{j=1}^{k} WG_j.$$

Since $WG_{j^*} = \max_{1\leq j\leq k} WG_j$, we get

$$OPT(\mathcal{S}) \leq \left(\frac{n}{\alpha^{k-1}} + 2\alpha \cdot k\right) \cdot WG_{j^*} \leq \left(\alpha + \frac{2\alpha}{\ln \alpha} \ln n\right) \cdot WG_{j^*}.$$

Thus,

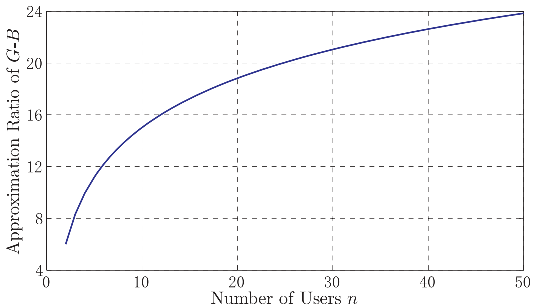

Fig. 3. Approximation ratio of Algorithm 2.

$$\frac{OPT(\mathcal{S})}{WG_{j^*}} \le \alpha + \frac{2\alpha}{\ln\alpha}\ln n. \tag{13}$$

Furthermore, we can specify α in terms of n so that the right side of (13) is minimized. Let

$$\frac{\partial\left(\alpha + \frac{2\alpha}{\ln\alpha}\ln n\right)}{\partial\alpha} = 0,$$

then we get $1 - \frac{2n}{(\ln\alpha)^2} + \frac{2n}{\ln\alpha} = 0$. Since $\alpha > 1$, we have $\alpha = \exp(\frac{2n}{n+\sqrt{2n+n^2}})$. Substituting this expression of α into (13), we finally get

$$\frac{OPT(\mathcal{S})}{WG_{j^*}} \le \exp\left(\frac{2\ln n}{\ln n + \sqrt{\ln n(2+\ln n)}}\right)\cdot$$
$$\left(1 + \ln n + \sqrt{\ln n(2+\ln n)}\right) \le 2e\cdot(\ln n + 1).$$

Since $\mathcal{S}$ is any instance of (6), we conclude that Algorithm 2 has an approximation ratio of $O(\ln n)$. □

Fig. 3 depicts the approximation ratio of Algorithm 2 for different numbers of active users. A cell typically has several tens of active users. In Fig. 3, we can find that the approximation ratio of Algorithm 2 increases slowly as the number of active users n grows.

5.4 A Word on Complexity

Algorithm 2 is based on an intuitive idea that utilizes a greedy method. We have shown that the gap between the optimal value and the value returned by the algorithm widens gradually as the number of active users n increases. Here, we show that Algorithm 2 has a small time complexity.

We denote by $\mathcal{N}$ the size of $A\times N$ and $\mathcal{N} = n\cdot(\binom{m}{2} + m)$. We use $\mathcal{N}_j$ to represent the size of S_j, $j\ge 1$. Note that $\sum_{j\ge 1}^{k}\mathcal{N}_j \le \mathcal{N}$.

Algorithm 2 first finds p_{max} and partitions $A\times N$ into $S_0, S_1, \ldots, S_k$, which requires a traversal of all elements of $A\times N$. This takes $O(\mathcal{N})$ time.

For each $S_j, j\ge 1$, Algorithm 2 calls the subroutine Algorithm 1. Algorithm 1 iteratively seeks an available (a,i) with smallest $tail(a)$, which usually requires a sort for $(a,i)\in S_j$ based on $tail(a)$. Thus, it can be shown that for S_j, invoking Algorithm 1 takes $O(\mathcal{N}_j\ln\mathcal{N}_j)$ time. Moreover,

$$\sum_{j=1}^{k} O(\mathcal{N}_j\ln\mathcal{N}_j) = \sum_{j=1}^{k} O(\mathcal{N}_j\ln\mathcal{N}) = O(\mathcal{N}\ln\mathcal{N}).$$

So, the total running time of Algorithm 2 can be calculated as

$$\begin{aligned} T_{G\text{-}B} &= O(\mathcal{N}) + \sum_{j=1}^{k} O(\mathcal{N}_j\ln\mathcal{N}_j) = O(\mathcal{N}\ln\mathcal{N}) \\ &= O\big(n\cdot m^2\ln(n\cdot m^2)\big) = O\big(n\cdot m^2(\ln n + \ln m)\big). \end{aligned} \tag{14}$$

6 A Local Ratio Technique-Based Algorithm

The basic idea of Algorithm 2 is intuitive; however, the performance is not so satisfying when the number of active users grows (Fig. 3). In this section, we introduce a more delicate approximation algorithm based on the local ratio technique [15], which achieves a constant approximation ratio of 2. The basic idea of local-ratio is similar to the *Dynamic Programming*, which aims at solving complex problems by breaking them down into simpler subproblems.

6.1 The Algorithm

The approximation algorithm is listed as Algorithm 3. It takes the number of RBs (m), the number of active users (n), and the profit function ($p: A\times N \to \mathbb{R}^{\ge 0}$) as the input. It outputs a feasible schedule S^* and its total profit W^*. Algorithm 3 first iterates from 1 through m to find candidate (a,i)s for S^*. In each loop, the algorithm tries to find the best candidate (a,i) (in the meaning of profit), and uses a stack S to store it. After the iteration, Algorithm 3 pops each (a,i) from S and adds (a,i) to S^* if it is valid. Finally, the algorithm generates a feasible schedule S^*.

Algorithm 3. A Local Ratio Technique Based Algorithm (L-R for short).

input m, n, p
$p' \leftarrow p$, $S \leftarrow \emptyset$ // S is a stack
for $j = 1$ to m **do**
  $(a^*, i^*) \leftarrow \arg\max_{(a,i)\in\{(a,i)|tail(a)=j\}}(p'(a,i))$ // break ties arbitrary
  if $p'(a^*, i^*) \le 0$ **then**
    continue
  end if
  $S.push((a^*, i^*))$
  for each (a,i) such that $i = i^*$ or a intersects a^* **do**
    if $p'(a,i) > 0$ **then**
      $p'(a,i) \leftarrow p'(a,i) - p'(a^*, i^*)$
    end if
  end for
end for
$S' \leftarrow \emptyset$
while $S \ne \emptyset$ **do**
  $(a,i) \leftarrow S.pop()$
  if $S' \cup \{(a,i)\}$ is a valid schedule **then**
    // a valid schedule means that $S' \cup \{(a,i)\}$ should meet the constraints of (6)
    $S' \leftarrow S' \cup \{(a,i)\}$
  end if
end while
$S^* \leftarrow S'$, $W^* \leftarrow \sum_{(a,i)\in S^*} p(a,i)$
return S^* and W^*

6.2 The Approximation Ratio

In this section, we will prove that Algorithm 3 has an approximation ratio of 2, and that this approximation ratio is tight. To obtain this result, we first introduce Lemma 5, which is an instance of the Local Ratio Theorem [15]. Then, we prove that Algorithm 3 is a 2-approximation algorithm for LTE UL FDPS. Finally, we use an example to show that the approximation ratio of 2 is tight.

Lemma 5. *Consider an uplink system which has m RBs and n active users. Let p, p_1, p_2 be profit functions such that $p(a,i) = p_1(a,i) + p_2(a,i), \forall (a,i) \in A \times N$. Let S^*, S_1^*, and S_2^* be optimal schedules for p, p_2, and p_3, respectively. Assume that S is a feasible schedule such that $r \cdot \sum_{(a,i)\in S} p_1(a,i) \geq \sum_{(a,i)\in S_1^*} p_1(a,i)$ and*

$$r \cdot \sum_{(a,i)\in S} p_2(a,i) \geq \sum_{(a,i)\in S_2^*} p_2(a,i),$$

is a constant. Then, we have

$$r \cdot \sum_{(a,i)\in S} p(a,i) \geq \sum_{(a,i)\in S^*} p(a,i).$$

Proof. Note that

$$\begin{aligned}\sum_{(a,i)\in S^*} p(a,i) &= \sum_{(a,i)\in S^*} p_1(a,i) + \sum_{(a,i)\in S^*} p_2(a,i)\\ &\leq \sum_{(a,i)\in S_1^*} p_1(a,i) + \sum_{(a,i)\in S_2^*} p_2(a,i).\end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_{(a,i)\in S_1^*} p_1(a,i) + \sum_{(a,i)\in S_2^*} p_2(a,i) &\leq r \cdot \sum_{(a,i)\in S} p_1(a,i)\\ &+ r \cdot \sum_{(a,i)\in S} p_2(a,i) = r \cdot \sum_{(a,i)\in S} p(a,i).\end{aligned}$$

Thus, the lemma is proved. □

Note that in Lemma 5, the profit functions p, p_1, and p_2 need not be nonnegative functions. That is to say, the profit function can take a negative value for some $(a,i) \in A \times N$. In Theorem 6, we will prove that Algorithm 3 has a constant approximation ratio of 2. The proof of this result relies on the observation that Lemma 5 applies to Algorithm 3 and we will use Lemma 5 inductively in the proof.

Theorem 6. *Algorithm 3 is a 2-approximation algorithm for LTE UL FDPS.*

Proof. We first introduce some necessary notations and definitions.

After the outermost for-loop is finished, the stack S can be represented as $\cup_{j=1}^{m} S_j$, where each $|S_j| \leq 1$. If in the jth for-loop, the algorithm adds some (a_j^*, i_j^*) to S, then $S_j = \{(a_j^*, i_j^*)\}$. Otherwise, $S_j = \emptyset$.

Correspondingly, the outermost for-loop iteratively generates a series of profit functions $(p_1^{(j)}, p_2^{(j)}), j = 1, \ldots, m$. For $j \geq 1$, if $S_j = \{(a_j^*, i_j^*)\}$,

$$p_1^{(j)}(a,i) = \begin{cases} p_2^{(j-1)}(a_j^*, i_j^*) \cdot 1_{\mathbb{R}^{>0}}\big(p_2^{(j-1)}(a,i)\big) & i = i_j^* \text{ or } a \text{ intersects } a_j^*,\\ 0 & \text{otherwise,}\end{cases}$$

and

$$p_2^{(j)}(a,i) = p_2^{(j-1)}(a,i) - p_1^{(j)}(a,i) \quad \forall (a,i) \in A \times N.$$

$1_{\mathbb{R}^{>0}}(x)$ is the characteristic function such that $1_{\mathbb{R}^{>0}}(x) = 1$ for $x > 0$, and $1_{\mathbb{R}^{>0}}(x) = 0$ for $x \leq 0$.

If $S_j = \emptyset$, we set $p_1^{(j)}(a,i) = 0$ and $p_2^{(j)}(a,i) = p_2^{(j-1)}(a,i)$. We let $p_2^{(0)}(a,i) = p(a,i)$ and

$$p_1^{(0)}(a,i) = 0, \forall (a,i) \in A \times N.$$

Thus, we have $p_1^{(j)}(a,i) + p_2^{(j)}(a,i) = p_2^{(j-1)}(a,i), j \geq 1$, $(a,i) \in A \times N$. According to the algorithm, it can be shown that for $j \geq 1$,

$$p_2^{(j)}(a,i) \leq 0, \forall tail(a) \leq j,$$

and that

$$\forall (a,i) \in A \times N, p_2^{(j)}(a,i) \geq p_2^{(k)}(a,i), j \leq k.$$

In addition, the while-loop equivalently generates a series of $S_j^*, j = 0, \ldots, m$, where $S_0^* = S^*$, $S_m^* = \emptyset$, and $S_j^* \subseteq \cup_{k=m}^{j+1} S_k$. S_j^* is regarded as the value of variable S' after the algorithm tries to add $\cup_{k=m}^{j+1} S_k$ to S'. So, it is obvious that

$$S_{j+1}^* \subseteq S_j^* \subseteq S_{j+1}^* \cup S_{j+1}, j = 0, \ldots, m-1.$$

We denote by $W_{opt}^{(j)}$ the total profit of the optimal schedule for $p_2^{(j)}$. If the optimal schedule is empty, we set $W_{opt}^{(j)} = 0$. In particular, $W_{opt}^{(0)}$ is the optimal value of (6). We define $W^{(j)} = \sum_{(a,i)\in S_j^*} p_2^{(j)}(a,i)$. If $S_j^* = \emptyset$, $W^{(j)} = 0$. In particular, $W^{(0)} = W^*$. So, what we are going to prove is $W_{opt}^{(0)} \leq 2W^{(0)} = 2W^*$.

In the following, we will prove by induction that $W_{opt}^{(j)} \leq 2W^{(j)}$, $j = m, m-1, \ldots, 1, 0$. When $j = 0$, we obtain the desired result. The mathematical induction starts from m down to 0.

The Basis. When $j = m$, $p^{(m)}(a,i) \leq 0$, $tail(a) \leq m$. That is, $p^{(m)}(a,i) \leq 0$, $\forall (a,i) \in A \times N$. So, $W_{opt}^{(m)} = 0$. Since $S_m^* = \emptyset$, $W_{opt}^{(m)} \leq 2W^{(m)}$ holds vacuously.

The Inductive Step. Assume that $W_{opt}^{(j)} \leq 2W^{(j)}$, $j \leq m$. we denote by $V_{opt}^{(j)}$ the total profit of the optimal schedule for $p_1^{(j)}$.

If $S_j = \emptyset$, then $p_2^{(j)}(a,i) = p_2^{(j-1)}(a,i)$, and $p_1^{(j)}(a,i) = 0$. So, $W_{opt}^{(j-1)} = W_{opt}^{(j)}$. Since $S_j^* \subseteq S_{j-1}^* \subseteq S_j^* \cup S_j$, $S_{j-1}^* = S_j^*$. Thus,

$$W^{(j-1)} = \sum_{(a,i)\in S_{j-1}^*} p_2^{(j-1)}(a,i) = \sum_{(a,i)\in S_j^*} p_2^{(j)}(a,i) = W^{(j)}.$$

Then, we get

$$W_{opt}^{(j-1)} = W_{opt}^{(j)} \leq 2W^{(j)} = 2W^{(j-1)}.$$

Otherwise, $S_j = \{(a_j^*, i_j^*)\}$. According to the algorithm, $p_2^{(j)}(a_j^*, i_j^*) = 0$. Because $S_j^* \subseteq S_{j-1}^* \subseteq S_j^* \cup S_j$,

$$\begin{aligned}2 \sum_{(a,i)\in S_{j-1}^*} p_2^{(j)}(a,i) &= 2 \sum_{(a,i)\in S_j^*} p_2^{(j)}(a,i)\\ &= 2W^{(j)} \geq W_{opt}^{(j)}.\end{aligned} \quad (15)$$

TABLE 1
An LTE UL FDPS Problem which Approaches the Approximation Ratio

(a,i)	$(\{1\},1)$	$(\{2\},1)$	$(\{1,2\},1)$
$p(a,i)$	1	1	1
(a,i)	$(\{1\},2)$	$(\{2\},2)$	$(\{1,2\},2)$
$p(a,i)$	$1-\epsilon$	0	1

On the other hand, S^*_{j-1} contains at least one element (a',i') such that $i'=i^*_j$ or a' intersects a. According to the algorithm, $\{(a',i')\}=S_k, k\geq j$. Thus,

$$p_2^{(j-1)}(a',i')\geq p_2^{(k-1)}(a',i')>0.$$

So, $p_1^{(j)}(a',i')=p_2^{(j-1)}(a^*_j,i^*_j)$. In this case,

$$\sum_{(a,i)\in S^*_{j-1}} p_1^{(j)}(a,i)\geq p_1^{(j)}(a',i')=p_2^{(j-1)}(a^*_j,i^*_j).$$

Then, we derive an upper bound for $V^{(j)}_{opt}$. We define $Supp_{p_1}=\{(a,i)\mid p_1^{(j)}(a,i)>0\}$. $Supp_{p_1}$ contains (a,i)s such that $i=i^*_j$ or a intersects a^*_j. Moreover, $p_2^{(j-1)}(a,i)\leq 0, tail(a)\leq j-1$, so for $(a,i)\in Supp_{p_1}$, $tail(a)\geq j$. Since $tail(a^*_j)=j$, $(a,i)\in Supp_{p_1}$ such that $i\neq i^*_j$ must intersects a^*_j at RB j. So, the optimal solution for $p_1^{(j)}$ at most have two (a,i)s $\in Supp_{p_1}$: for one (a,i), $i=i^*_j$, and for another (a,i), a intersects a^*_j. That is, $V^{(j)}_{opt}\leq 2p_2^{(j-1)}(a^*_j,i^*_j)$. Thus,

$$2\sum_{(a,i)\in S^*_{j-1}} p_1^{(j)}(a,i)\geq 2p_2^{(j-1)}(a^*_j,i^*_j)\geq V^{(j)}_{opt}. \tag{16}$$

Since $p_2^{(j-1)}(a,i)=p_1^{(j)}(a,i)+p_2^{(j)}(a,i)$, according to (15) and (16), Lemma 5 indicates that

$$2W^{(j-1)}=2\sum_{(a,i)\in S^*_{j-1}} p_2^{(j-1)}(a,i)\geq W^{(j-1)}_{opt}.$$

Since both the basis and the inductive step have been proved, it has now been proved by mathematical induction that $W^{(0)}_{opt}\leq 2W^*$. So, Algorithm 3 has an approximation ratio of 2. □

Moreover, we find an LTE UL FDPS problem that shows that the approximation ratio of Algorithm 3 is tight.

Proposition 1. *The approximation ratio of 2 is tight.*

Proof. In the LTE UL FDPS problem of Table 1, $m=n=2$, and $0<\epsilon<1$. The optimal solution is $\{(\{2\},1),(\{1\},2)\}$, and the optimal value is $2-\epsilon$. However, for this instance of LTE UL FDPS, Algorithm 3 returns $S^*=\{(\{1\},1)\}$ and $W^*=1$. So, the approximation ratio $\geq\frac{2-\epsilon}{1},\forall 0<\epsilon<1$. When $\epsilon\to 0$, $2-\epsilon\to 2$. Thus, the approximation ratio is tight. □

6.3 A Word on Complexity

We have shown that Algorithm 3 has a constant approximation ratio, and here we analyze the complexity of the algorithm. In the jth for loop, the number of (a,i)s such that $tail(a)=j$ is $m\cdot j$. So, to obtain (a^*_j,i^*_j), the algorithm at least accesses $m\cdot j$ (a,i)s. In addition, since in the jth for loop, $p'(a,i)\leq 0$, $tail(a)\leq j-1$, the number of (a,i)s such that $p'(a,i)$ is changed is at least $\binom{m}{2}+m-\frac{j(j-1)}{2}+(n-1)\cdot j\cdot(m-j+1)$. So, the outermost for loop takes

$$\begin{aligned}&\sum_{j=1}^{m}\left(m\cdot j+\binom{m}{2}+m+(n-1)\cdot j\cdot(m-j+1)-\frac{j(j-1)}{2}\right)\\&=\frac{1}{6}\left(-m+m^3+5mn+6m^2n+m^3n\right)\\&=O(m^3n)\end{aligned}$$

running time. Since S contains at most m elements, the while loop takes $O(m)$ time. Thus, the running time of Algorithm 3 is

$$T_{L\text{-}R}=O(m^3n). \tag{17}$$

7 Stability Analysis

The assumption of an infinitely backlogged model in scheduling analysis is conventional in many literatures. Nevertheless, in real networks, packets are generated for each user in accordance with a stochastic arrival process with respect to applications, which may result in the instability of queues. Therefore, it is preferable to take system utility maximization and queuing stability into account simultaneously when we design scheduling algorithms. Since the L-R Algorithm already achieves a fairly good performance, we now investigate its stability. In this section, we formalize a specific LTE UL FDPS problem, which is a special case of the general problem (6), and the *L-R Algorithm* can also be applied with the same approximation ratio of 2. Similar to the methodology used in [14], we utilize the *Lyapunov Drift* to prove the queuing stability of the L-R Algorithm in LTE UL system.

7.1 A Specific Problem

We consider a specific problem that incorporates the queue length and channel quality information in the LTE UL. In this case, we allocate each set of *contiguous* RBs $a\in A$ to user i to maximize $\sum_{i\in N}\sum_{a\in A}\sum_{a:c\in a}Q_iX^a_iR^c_i$ in each TTI. The new optimization problem is formalized as follows:

$$\begin{aligned}&\max\quad \sum_{i\in N}\sum_{a\in A}\sum_{a:c\in a}Q_iX^a_iR^c_i,\\&\text{subject to}:\\&\quad\text{for each RB } c\in M:\quad \sum_{i\in N,a:c\in a}X^a_i\leq 1,\\&\quad\text{for each user } i\in N:\quad \sum_{a\in A}X^a_i\leq 1,\\&\quad\forall i\in N,\forall a\in A:\quad X^a_i\in\{0,1\}.\end{aligned} \tag{18}$$

To solve the problem (18), we actually intend to find the most advantageous way to assign an $a\in A$ to user i so as to maximize the throughput and keep the queue lengths bounded. The specific problem can be also considered as a multicarrier variant of the well-known *MaxWeight Match* problem. The first constraint of (18) shows that every RB is assigned to at most one user. The second constraint ensures that each user can get no more than one set of *contiguous* RBs. Especially, for any user i, $\sum_{a\in A}X^a_i=0$ means the user will have no chance to send data in this TTI. Now, we prove the L-R Algorithm solves (18) with the same approximation ratio of 2 as it does in (6).

Lemma 6. *The* L-R Algorithm *also solves the specific LTE UL FDPS problem (18) in a polynomial runtime with a constant approximation ratio of 2.*

Proof. We can utilize the profit function to express the objective of (18) like (3):

$$p(a, i) = Q_i \sum_{a:c \in a} R_i^c.$$

Here, the profit $p(a, i)$ is obtained as long as the contiguous set of RBs a is assigned to user i. In other words, the profit value exactly equals to the product of queue length Q_i and the total data rate $\sum_{a:c\in a} R_i^c$. Evidently, the LTE UL FDPS problem (18) is a special case of (6). The *L-R Algorithm* can also be applied here to solve it with the same approximation performance, and thus we prove the lemma. □

7.2 Necessary Definitions

Before we conduct the stability analysis, the necessary definitions are presented. The definition of (ω, ϵ)-admissible for single-carrier system is introduced in [27]. We extend that definition to the LTE UL multicarrier scenario.

Definition 2. *We say that a system is (ω, ϵ)-admissible if the adversary has a valid schedule set of $Y_i^a(t) \in \{0, 1\}$ such that in any window $[t_0, t_0 + \omega - 1]$, for the arrival process $A_i(t')$ and channel rate $R_i^c(t)$ we have*

$$\sum_{t'=t_0}^{t_0+\omega-1} A_i(t') \leq (1-\epsilon) \sum_{t=t_0}^{t_0+\omega-1} \sum_{a \in A} \sum_{a:c \in a} Y_i^a(t) R_i^c(t) \quad \forall i, \tag{19}$$

where $\omega \in Z^+$, $\epsilon < 1$, and $\sum_{i,a:c\in a} Y_i^a(t) = 1, \forall t, i, c$.

Informally, an algorithm is said to be stable if it keeps the queue sizes bounded whenever this is achievable [14]. More formally, we define the queuing stability of scheduling algorithm for LTE uplink as follows:

Definition 3. *An scheduling algorithm for LTE uplink is stable if it keeps the queue lengths bounded for any (ω, ϵ)-admissible systems.*

Suppose that a feasible scheduler Y ensures the arrival process A and channel rate R satisfy the relation of (19), the queue will not increase steeply and its length can be bounded. Accordingly, the (ω, ϵ)-admissible explicitly describes the basic propositions of some LTE UL systems, which will be stably scheduled by the feasible FDPS algorithms.

The procedure of stability proof follows a standard technique utilized in [14]. In this paper, the procedure is called *Lyapunov drift analysis*, which is similarly conducted in [28] and [25] as well. The fundamental analysis consists in showing the *Lyapunov function* has a *negative drift* when the queue lengths are sufficiently large. Here, the Lyapunov drift is computed by $L(t+\omega) - L(t)$, where $L(t) = \sum_i (Q_i(t))^2$ is the Lyapunov function. For reference simplified, we use some notations to represent different algorithms in the following contents. Namely, X, X^*, and Y denote the *L-R Algorithm*, the *Optimal Algorithm* and any other Algorithm for Problem (18), respectively. More specifically, X, X^*, and Y comprise the Boolean variables X_i^a, X_i^{*a} and Y_i^a, respectively.

To investigate the stable conditions for the *L-R Algorithm* X, it is acceptable to assume that the *Optimal Algorithm* X^* is stable for any (ω, ϵ)-admissible LTE UL system, where $\omega \in Z^+$, $\epsilon < 1$. Afterwards, we figure out the propositions of (ω_0, ϵ_0)-admissible LTE UL systems, which will be stably scheduled by the *L-R Algorithm* X. Moreover, the parameters (ω_0, ϵ_0) are calculated and expressed by (ω, ϵ), which are supposed as known parameters.

7.3 Stability Analysis

Now, we perform the Lyapunov drift analysis for the *L-R Algorithm*, which approximately solves the LTE UL FDPS problem (6) and (18). Some necessary and reasonable assumptions for LTE UL system used in the stability analysis are listed below:

1. The channel rates are bounded in a finite set, i.e., R^{sup} is the supremum of these rates while R^{inf} is the infimum.
2. The arrival process is also bounded, i.e., A^{sup} is the supremum while A^{inf} is the infimum.
3. The queue sizes among all the n users within some time interval $(t_0, t_0 + \omega - 1)$ is bounded, i.e., Q^{sup} is the supremum while Q^{inf} is the infimum.

The stability conclusions with the values of (ω_0, ϵ_0) are demonstrated in Theorem 7.

Theorem 7. *The* L-R Algorithm *X is stable for any (ω_0, ϵ_0)-admissible LTE UL systems as long as $\forall i, c$, the user rates $R_i^c(t)$ cannot be zero for arbitrarily long periods, where*

$$\frac{1}{2} \leq \epsilon_0 \leq 1 + \frac{(n-1)A^{inf}}{B_4} - \frac{2(1-\epsilon)B_3 Q^{sup}}{B_4 Q^{inf}}, \tag{20}$$

and

$$\omega_0 \leq \omega. \tag{21}$$

Proof. It takes two steps to prove Theorem 7. We start with showing that in any (ω_0, ϵ_0)-admissible LTE UL systems, once the queue lengths are sufficiently large then the *Lyapunov function* has a negative drift. Afterwards, we suppose that the parameters (ω, ϵ) are already known for the stable X^*, and hence (ω_0, ϵ_0) of X can be calculated and expressed by them.

First of all, the fundamental relation among queue lengths in different TTI yields (22), where $[x]^+$ denotes $max(0, x)$,

$$\begin{aligned} Q_i(t_0 + \omega_0) = \Bigg[& Q_i(t_0) + \sum_{t=t_0}^{t_0+\omega_0-1} A_i(t) \\ & - \sum_{t=t_0}^{t_0+\omega_0-1} \sum_{a \in A} \sum_{a:c \in a} X_i^a(t) R_i^c(t) \Bigg]^+ \quad \forall i \in N, \end{aligned} \tag{22}$$

which can be rewritten as

$$Q_i(t_0+\omega_0) \geq Q_i(t_0) + \sum_{t=t_0}^{t_0+\omega_0-1} A_i(t) - \sum_{t=t_0}^{t_0+\omega_0-1} \sum_{a\in A} \sum_{a:c\in a} X_i^a(t) R_i^c(t) \quad \forall i \in N. \tag{23}$$

To simplify the expressions, the abbreviations of notations $\sum_i$, $\sum_{a,c}$, and $\sum_t$ stand for $\sum_{i\in N}$, $\sum_{a\in A}\sum_{a:c\in a}$, and $\sum_{t=t_0}^{t_0+\omega_0-1}$, respectively. Then, (23) is substituted into the Lyapunov drift and we have

$$\begin{aligned} L(t_0+\omega_0) - L(t_0) &= \sum_i Q_i(t_0+\omega_0)^2 - \sum_i Q_i(t_0)^2 \\ &= 2\sum_i \Big(\sum_t A_i(t)\Big)^2 + 2\sum_i \Big(\sum_t \sum_{a,c} X_i^a(t)R_i^c(t)\Big)^2 \\ &\quad + 2\sum_i Q_i(t_0)\Big(\sum_t A_i(t) - \sum_t\sum_{a,c} X_i^a(t)R_i^c(t)\Big) \\ &\quad - \sum_i \Big(\sum_t A_i(t)\Big)^2 - \sum_i \Big(\sum_t\sum_{a,c} X_i^a(t)R_i^c(t)\Big)^2 \\ &\quad - 2\sum_i\sum_t A_i(t)\Big(\sum_t\sum_{a,c} X_i^a(t)R_i^c(t)\Big), \end{aligned}$$

where the last three terms form a perfect square, and hence we obtain

$$\begin{aligned} L(t_0+\omega_0) - L(t_0) &\leq 2\sum_i \Big(\sum_t A_i(t)\Big)^2 + 2\sum_i \Big(\sum_t\sum_{a,c} X_i^a(t)R_i^c(t)\Big)^2 \\ &\quad + 2\sum_i Q_i(t_0)\Big(\sum_t A_i(t) - \sum_t\sum_{a,c} X_i^a(t)R_i^c(t)\Big). \end{aligned} \tag{24}$$

Since the system is (ω_0, ϵ_0)-admissible, the total amount of arriving data for each user is upper bounded (see (19)) by a function of R^{sup}, m, ω_0, and ϵ_0. In particular,

$$\begin{aligned} \sum_{t=t_0}^{t_0+\omega_0-1} A_i(t) &\leq (1-\epsilon_0)\sum_{t=t_0}^{t_0+\omega_0-1}\sum_{a,c} X_i^a(t)R_i^c(t) \\ &\leq (1-\epsilon_0)\omega_0 m R^{sup}. \end{aligned}$$

Consequently, the first term of the right side of (24) has an upper bound while the second term is also bounded by a function of R^{sup}, m, ω_0, n, and ϵ_0. We denote B_1 as the upper bound of the first two terms, i.e.,

$$B_1 = 2n((1-\epsilon_0)^2+1)(\omega_0 m R^{sup})^2.$$

As for the third term, provided that $t_0 \leq t \leq t_0+\omega_0$, the following inequality holds obviously:

$$Q_i(t_0) \leq Q_i(t) + \omega_0 m R^{sup}.$$

Hence, we have

$$\begin{aligned} L(t_0+\omega_0) - L(t_0) &\leq B_1 + 2\omega_0 m R^{sup}\sum_i\sum_t\Big(A_i(t) - \sum_{a,c} X_i^a(t)R_i^c(t)\Big) \\ &\quad + 2\sum_i Q_i(t)\Big(\sum_t A_i(t) - \sum_t\sum_{a,c} X_i^a(t)R_i^c(t)\Big). \end{aligned} \tag{25}$$

Similarly, the second term of the above expression can be upper bounded by a function of R^{sup}, m, n, and ω_0, which equals to B_2:

$$B_2 = -2n\epsilon_0(\omega_0 m R^{sup})^2.$$

Besides, the objective values of (18) under different algorithms form the succeeding inequality:

$$\begin{aligned} 2\sum_{t,i,a,c} Q_i(t)X_i^a(t)R_i^c(t) &\geq \sum_{t,i,a,c} Q_i(t){X^*}_i^a(t)R_i^c(t) \\ &\geq \sum_{t,i,a,c} Q_i(t)Y_i^a(t)R_i^c(t). \end{aligned} \tag{26}$$

Again, both (ω_0, ϵ_0)-admissible proposition and (26) are applied together. In consequence, (25) can be rewritten as

$$\begin{aligned} &L(t_0+\omega_0) - L(t_0) \\ &\leq B_1 + B_2 + 2\sum_i Q_i(t)\Big(\sum_t A_i(t) - \sum_t\sum_{a,c} X_i^a(t)R_i^c(t)\Big) \\ &\leq B_1 + B_2 + 2\sum_i\sum_t\sum_{a,c} Q_i(t)R_i^c(t) \\ &\quad \big((1-\epsilon_0)Y_i^a(t) - X_i^a(t)\big) \\ &\leq B_1 + B_2 + 2\sum_{i,t,a,c}(1-2\epsilon_0)Q_i(t)X_i^a(t)R_i^c(t). \end{aligned} \tag{27}$$

Evidently, the parameter ϵ_0 needs to satisfy (20) (i.e., $\frac{1}{2} \leq \epsilon_0$) to keep the third term of (27) nonpositive. Suppose $j = argmin_i Q_i(t)$ and $Q_{min}(t) = Q_j(t)$, then (28) is hence derived:

$$\begin{aligned} L(t_0+\omega_0) - L(t_0) &\leq B_1 + B_2 + 2(1-2\epsilon_0)\sum_{i,t,a,c} Q_i(t)X_i^a(t)R_i^c(t) \\ &\leq B_1 + B_2 + 2(1-2\epsilon_0)R^{inf} m \sum_t Q_{min}(t). \end{aligned} \tag{28}$$

Once the queue lengths are sufficiently large, the potential function $L(t_0)$ decreases, which implies the stability of the *L-R Algorithm X*. So far the stability is proved, we turn to compute the key parameters (ω_0, ϵ_0), and use (ω, ϵ) of X^* to express them.

Obviously, when $\epsilon < 1$, inequality (26) yields

$$\begin{aligned} 2(1-\epsilon)\sum_{t,i,a,c} Q_i(t)X_i^a(t)R_i^c(t) \\ \geq (1-\epsilon)\sum_{t,i,a,c} Q_i(t){X^*}_i^a(t)R_i^c(t). \end{aligned} \tag{29}$$

Combining the (ω, ϵ)-admissible proposition for the stable X^*, the bounded queue lengths and (29), we have

$$\begin{aligned} 2(1-\epsilon)Q^{sup}\sum_{t,i,a,c} X_i^a(t)R_i^c(t) &\geq 2(1-\epsilon)\sum_{t,i,a,c} Q_i(t)X_i^a(t)R_i^c(t) \\ &\geq (1-\epsilon)\sum_{t,i,a,c} Q_i(t){X^*}_i^a(t)R_i^c(t) \\ &\geq Q^{inf}\sum_{t,i,a,c} A_i(t). \end{aligned}$$

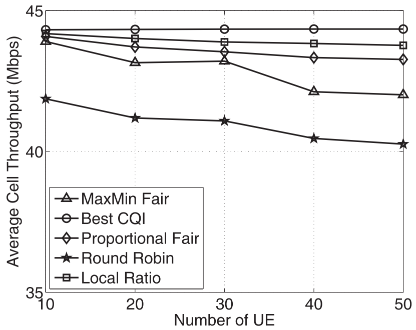

Fig. 4. The average throughput of all the UEs. (User speed is 5 km/h.)

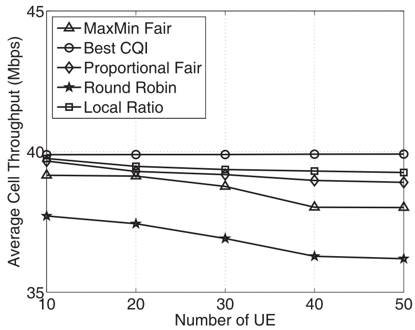

Fig. 5. The average throughput of all the UEs. (User speed is 50 km/h.)

Likewise, $\sum_{t,i,a,c} X_i^a(t)R_i^c(t)$ can be upper bounded by a function (denoted by B_3) of m, n, ω, and R^{sup}. $\forall i \in N$, the total arriving data of user i has a lower bound, and thus we obtain

$$\begin{aligned} 2(1-\epsilon)Q^{sup}B_3 &\geq Q^{inf}\sum_{t,i,a,c} A_i(t) \\ &\geq Q^{inf}\sum_{t=t_0}^{t=t_0+\omega-1} A_i(t) + Q^{inf}(n-1)A^{inf}. \end{aligned} \quad (30)$$

As long as we set $\omega_0 \leq \omega$ (see (21)), (30) can be rewritten as follows:

$$\begin{aligned} \sum_{t=t_0}^{t=t_0+\omega_0-1} A_i(t) &\leq \sum_{t=t_0}^{t=t_0+\omega-1} A_i(t) \\ &\leq 2(1-\epsilon)B_3\frac{Q^{sup}}{Q^{inf}} - (n-1)A^{inf}. \end{aligned}$$

So far, we find out an upper bound for $\sum_{t=t_0}^{t=t_0+\omega_0-1} A_i(t)$. To guarantee the LTE UL system is (ω_0, ϵ_0)-admissible so that X schedules stably, we require (19) holds for (ω_0, ϵ_0), i.e.,

$$\sum_{t=t_0}^{t_0+\omega_0-1} A_i(t) \leq (1-\epsilon_0)\sum_{t=t_0}^{t_0+\omega_0-1}\sum_{a\in A}\sum_{a:c\in a} X_i^a(t)R_i^c(t) \quad \forall i. \quad (31)$$

Moreover, the right part of (31) can be intuitively lower bounded by $(1-\epsilon_0)\omega_0 mR^{inf}$. We use B_4 to represent $\omega_0 mR^{inf}$. Consequently, (31) holds provided that

$$2(1-\epsilon)B_3\frac{Q^{sup}}{Q^{inf}} - (n-1)A^{inf} \leq (1-\epsilon_0)\omega_0 mR^{inf},$$

where ϵ_0 can be eventually derived as

$$\epsilon_0 \leq 1 + \frac{(n-1)A^{inf}}{B_4} - \frac{2(1-\epsilon)B_3Q^{sup}}{B_4Q^{inf}}.$$

In other words, the LTE UL system can be considered as (ω_0, ϵ_0)-admissible in case (ω_0, ϵ_0) satisfy (20) and (21), respectively. In this kind of cellular systems, the *L-R Algorithm* schedules stably. Hence, we prove the theorem and complete the stability analysis. □

8 Performance Evaluation

In this section, we utilize the LTE UL simulator [29], which is an open platform for academic research. On that platform, we deploy the Algorithm 3 and perform simulations for the Local Ratio (L-R) scheduler (Algorithm 3) as well as other well-known schedulers, including Max-Min fairness, Best CQI, Proportional-Fair and Round Robin (RR). To implement the Algorithm 3, the profit function is specified as the current data rate, namely,

$$p(a,i) = \sum_{c\in a} R_i^c,$$

where R_i^c is the average data rate of user i on RB c within the present TTI.

We carry out five similar simulations and calculate the average *cell throughput*[4] for each scheduler. The results are provided with the average values of a 10,000 TTIs duration for each number of UEs. The operational frequency is 2 GHz while the frequency bandwidth is 20 MHz. Two kinds of user speed are set, i.e., the walking mode (5 km/h) and the vehicular mode (50 km/h). As illustrated in Figs. 4 and 5, the RR and Best CQI (that always schedules user with the best channel quality) algorithms are the lower and upper bounds of all schedulers in terms of system performance (represented by cell throughput in this case), respectively. Along with the increasing number of users, the average throughput of all schedulers trends to decrease slightly. The throughput of the L-R scheduler is between the PF and the upper bound (Fig. 4), which indicates its good performance. When the user speed is accelerated to 50 km/h, due to serious channel fading, the average cell throughput of all schedulers are degraded (Fig. 5). However, in this mode, the L-R scheduler still achieves better performance than the PF scheduler.

To investigate the stability of the L-R scheduler, we compute a stability region based on Theorem 7. It is assumed that the Optimal Algorithm for problem (6) is stable for any (1-0)-admissible LTE system, namely $\omega = 1$, $\epsilon = \frac{1}{2}$. In this case, to hold (20) and (21), we consider $\frac{1}{2} \leq 1 + \frac{(n-1)A^{inf}}{B_4} - \frac{B_3Q^{sup}}{B_4Q^{inf}}$ and $\omega_0 = 1$. In other words, the 2-approximation *L-R Algorithm* is stable as long as Q^{sup} is set,

$$Q^{sup} \leq \frac{B_4}{2B_3} + \frac{(n-1)A^{inf}}{B_3}. \quad (32)$$

Furthermore, we present simulation results of the average queue length of the L-R scheduler when different arrival rates are set. The queue length are sampled and

4. The cell throughput is the total throughput of all the uploading UEs that a eNodeB covers.

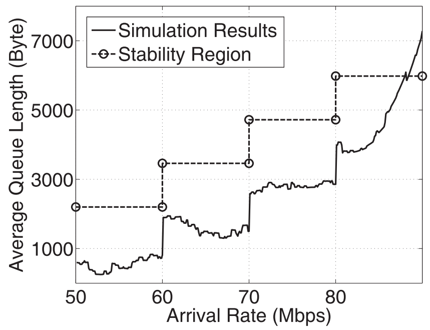

Fig. 6. The average queue length of the Local Ratio scheduler and its stability region.

averaged in a period of 100 TTIs, i.e., 0.1 s. The arrival rates A are also set 50/60/70/80 Mbps. Each rate lasts for 10,000 TTIs. According to (32), we can plot a stability region for the L-R scheduler. Theoretically, the queue length will reside in this region without overflow when a certain arrival rate is set for the L-R scheduler. The stability region (the dash line) and the simulation results (the solid line) are combined in Fig. 6. It indicates that the queues are stable when the arrival rates are relatively small. However, when the arrival rate is 80 Mbps, the average queue length is increasing steeply and exceeds the stability region, which implies overflow and the instability of the scheduling algorithm. In other words, the L-R Algorithm is stable for LTE UL system within a certain admissible region of the arrival rate.

9 Conclusion

In this paper, we consider a general FDPS problem for the LTE uplink. Our formulation of this problem applies to many scheduling policies. We prove that the LTE UL FDPS problem is MAX SNP-hard, which implies that the problem is hard to approximate. We propose two approximation algorithms for this scheduling problem, both of which are computable in polynomial time. The first algorithm is based on a simple greedy method, and it has an approximation ratio of $O(\ln n)$, where n is the number of active users in the cell. The second *L-R Algorithm* is more delicate, which though achieves a constant approximation ratio of 2.

Furthermore, we perform *Lyapunov drift analysis* to investigate the stability of the *L-R Algorithm*. We first specify the general profit function to incorporate queue length and channel quality information. The specific problem (18) is a special case of (6). Accordingly, the L-R Algorithm can be also applied with the same approximation ratio of 2. Subsequently, we show the Lyapunov drift for problem (18) is negative once the queue lengths are sufficiently large. Finally, the L-R Algorithm is proved to be stable for any (ω_0, ϵ_0)-admissible LTE UL system. The simulation results indicate good performance of the Local Ratio scheduler.

Acknowledgments

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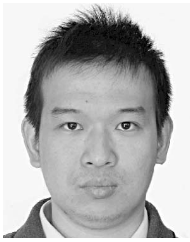

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